

Déterminants : Formule de Laplace

Solutions pour la série 8

Exercice 1

$$\text{a) } |\mathbf{A}| = \begin{vmatrix} 4 & 5 \\ -3 & -2 \end{vmatrix} = 4 \cdot (-2) - (-3) \cdot 5 = -8 + 15 = 7$$

$$\text{b) } |\mathbf{B}| = \begin{vmatrix} 3 & -4 \\ 6 & -8 \end{vmatrix} = 3 \cdot (-8) - 6 \cdot (-4) = -24 + 24 = 0$$

$$\text{c) } |\mathbf{C}| = \begin{vmatrix} a-b & a \\ a & a+b \end{vmatrix} = (a-b)(a+b) - a^2 = -b^2$$

Exercice 2

$$\text{a) } |\mathbf{A}| = \begin{vmatrix} 1 & -3 \\ -2 & 4 \end{vmatrix} = 1 \cdot 4 - (-2) \cdot (-3) = 4 - 6 = -2$$

$$\text{b) } |\mathbf{B}| = \begin{vmatrix} 1/2 & -3/2 \\ -2/3 & 4/3 \end{vmatrix} = \frac{1}{2} \cdot \frac{4}{3} - \left(-\frac{2}{3}\right) \cdot \left(-\frac{3}{2}\right) = \frac{2}{3} - 1 = -\frac{1}{3}$$

$$\text{c) } |\mathbf{C}| = \left| \frac{1}{4} \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \right| = \begin{vmatrix} 1/4 & -3/4 \\ -1/2 & 1 \end{vmatrix} = \frac{1}{4} \cdot 1 - \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{4}\right) = \frac{1}{4} - \frac{3}{8} = -\frac{1}{8}$$

$$\text{REMARQUE. On a } \mathbf{C} = \frac{1}{4}\mathbf{A} \text{ et } |\mathbf{C}| = \left(\frac{1}{4}\right)^2 |\mathbf{A}| = \frac{1}{16} \cdot (-2) = -\frac{1}{8}.$$

Exercice 3

$$\begin{aligned} \text{a) } |\mathbf{A}| &= \begin{vmatrix} 1 & -3 & 3 \\ 4 & 2 & 0 \\ -2 & -7 & 5 \end{vmatrix} \begin{matrix} \textcolor{red}{1} & \textcolor{red}{-3} \\ \textcolor{red}{4} & \textcolor{red}{2} \\ \textcolor{red}{-2} & \textcolor{red}{-7} \end{matrix} = (1 \cdot 2 \cdot 5) + ((-3) \cdot 0 \cdot (-2)) + (3 \cdot 4 \cdot (-7)) \\ &\quad - (3 \cdot 2 \cdot (-2)) - (1 \cdot (-7) \cdot 0) - ((-3) \cdot 4 \cdot 5) \\ &= 10 + 0 - 84 + 12 - 0 + 60 = -2 \end{aligned}$$

$$\begin{aligned} \text{b) } |\mathbf{B}| &= \begin{vmatrix} 1 & -1 & -3 \\ 2 & 2 & -2 \\ 3 & 2 & -4 \end{vmatrix} \begin{matrix} \textcolor{red}{1} & \textcolor{red}{-1} & \textcolor{red}{-3} \\ \textcolor{red}{2} & \textcolor{red}{2} & \textcolor{red}{-2} \end{matrix} = (1 \cdot 2 \cdot (-4)) + (2 \cdot 2 \cdot (-3)) + (3 \cdot (-1) \cdot (-2)) \\ &\quad - ((-3) \cdot 2 \cdot 3) - ((-2) \cdot 2 \cdot 1) - ((-4) \cdot (-1) \cdot 2) \\ &= -8 - 12 + 6 + 18 + 4 - 8 = 0 \end{aligned}$$

$$\begin{aligned} \text{c) } |\mathbf{C}| &= \begin{vmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 1 & -2 \end{vmatrix} = ((-1) \cdot 5 \cdot (-2)) + (2 \cdot 1 \cdot 1) + (3 \cdot 3 \cdot 0) \\ &\quad - (1 \cdot 5 \cdot 3) - (0 \cdot 1 \cdot (-1)) - ((-2) \cdot 3 \cdot 2) \\ &= 10 + 2 + 0 - 15 - 0 + 12 = 9 \end{aligned}$$

Exercice 4

a) On développe le déterminant selon la première colonne (ou ligne) :

$$|\mathbf{A}| = \begin{vmatrix} \textcolor{red}{0} & 0 & 2 \\ \textcolor{red}{0} & 7 & -1 \\ \textcolor{red}{-2} & -5 & 3 \end{vmatrix} = -2 \cdot \begin{vmatrix} 0 & 2 \\ 7 & -1 \end{vmatrix} = -2 \cdot (-14) = 28$$

b) On développe le déterminant selon la deuxième colonne (ou la troisième ligne) :

$$|\mathbf{B}| = \begin{vmatrix} \cos(\alpha) & \mathbf{0} & \sin(\alpha) \\ \sin(\alpha) & \mathbf{0} & \cos(\alpha) \\ 0 & \mathbf{1} & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{vmatrix} = -(\cos^2(\alpha) - \sin^2(\alpha)) = -\cos(2\alpha)$$

Exercice 5

a) La matrice est triangulaire (supérieure), son déterminant est simplement le produit de ses éléments diagonaux :

$$|\mathbf{A}| = \begin{vmatrix} -3 & -4 & 1 & 2 \\ 0 & 5 & -5 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -7 \end{vmatrix} = (-3) \cdot 5 \cdot 2 \cdot (-7) = 210$$

$$\begin{aligned} \text{b) } |\mathbf{B}| &= \begin{vmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{5} \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} = 5 \cdot \begin{vmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{4} \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 5 \cdot (-4) \cdot \begin{vmatrix} \mathbf{0} & \mathbf{0} & \mathbf{3} \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} \\ &= 5 \cdot (-4) \cdot 3 \cdot \begin{vmatrix} \mathbf{0} & \mathbf{2} \\ 1 & 0 \end{vmatrix} = 5 \cdot (-4) \cdot 3 \cdot (-2) \cdot 1 = 5! = 120 \end{aligned}$$

$$\begin{aligned} \text{c) } |\mathbf{C}| &= \begin{vmatrix} 1 & 0 & 2 & 5 \\ -2 & 1 & 5 & 0 \\ 0 & -1 & 0 & 3 \\ \mathbf{0} & \mathbf{0} & \mathbf{-3} & \mathbf{0} \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & \mathbf{0} & 5 \\ -2 & \mathbf{1} & 0 \\ 0 & \mathbf{-1} & 3 \end{vmatrix} = 3 \left(\begin{vmatrix} 1 & 5 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 5 \\ -2 & 0 \end{vmatrix} \right) \\ &= 3(3 + 10) = 39 \end{aligned}$$

$$\begin{aligned} \text{d) } |\mathbf{D}| &= \begin{vmatrix} -3 & 8 & 1 & -2 & 5 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 2 \\ 6 & 0 & 0 & 4 & 5 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{0} \end{vmatrix} = -2 \cdot \begin{vmatrix} -3 & \mathbf{8} & 1 & 5 \\ 1 & \mathbf{0} & 0 & 1 \\ 0 & \mathbf{0} & -1 & 2 \\ 6 & \mathbf{0} & 0 & 5 \end{vmatrix} = -2 \cdot (-8) \cdot \begin{vmatrix} 1 & \mathbf{0} & 1 \\ 0 & \mathbf{-1} & 2 \\ 6 & \mathbf{0} & 5 \end{vmatrix} \\ &= -2 \cdot (-8) \cdot (-1) \cdot \begin{vmatrix} 1 & 1 \\ 6 & 5 \end{vmatrix} = -16 \cdot (5 - 6) = 16 \end{aligned}$$

Exercice 6

On a

$$\begin{vmatrix} 2 & 1 & 1 \\ x & -3 & -x \\ 4 & 3x & 2 \end{vmatrix} = -12 + 3x^2 - 4x - (-12 - 6x^2 + 2x) = 9x^2 - 6x$$

et

$$\begin{vmatrix} 2 & 1 & 1 \\ x & -3 & -x \\ 4 & 3x & 2 \end{vmatrix} = 1 \iff 9x^2 - 6x = 1 \iff x_{1,2} = \frac{6 \pm \sqrt{36 + 36}}{18} = \frac{6 \pm 6\sqrt{2}}{18}.$$

Les deux solutions de l'équation sont donc $x_{1,2} = \frac{1 \pm \sqrt{2}}{3}$.